

Manifolds, Tangent spaces and Differentials

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Exercise 1 (The projective space). Let \mathbb{RP}^n be the space defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation “belonging to the same vector line”. For $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we denote by $[x_0 : \dots : x_n]$ its class in \mathbb{RP}^n .

1. Show that \mathbb{RP}^n is compact Hausdorff and that the canonical projection $p : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ is open.
2. Let $i \in \{0, \dots, n\}$. Show that $U_i = \{[x_0 : \dots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$ is open in \mathbb{RP}^n , and construct a homeomorphism $\varphi_i : U_i \rightarrow \mathbb{R}^n$.
3. Show that \mathbb{RP}^n equipped with the atlas $(\varphi_i)_{i \in \{0, \dots, n\}}$ is a smooth manifold.
4. Show that p is smooth and that a map $f : \mathbb{RP}^n \rightarrow M$ is smooth if and only if $f \circ p$ is smooth, given any smooth manifold M .
5. Show that the restriction of p to \mathbb{S}^n is a local diffeomorphism.
6. Show that \mathbb{RP}^1 is diffeomorphic to \mathbb{S}^1 .
7. Let $A \in GL_{n+1}(\mathbb{R})$, we denote by $h_A : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$ its induced map (why is it well defined?). Show that h_A is a diffeomorphism and give its differential.
8. Given $A \in GL_{n+1}(\mathbb{R})$, what are the fixed points of h_A ?

Exercise 2 (Tangent space of a submanifold). Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be submanifolds of \mathbb{R}^m and \mathbb{R}^n respectively

1. Describe the tangent space $T_p M \subset \mathbb{R}^m$ of the submanifold M of \mathbb{R}^m at a point p , for each of the four characterizations of a submanifold. Why is there no ambiguity in identifying the tangent space $\subset \mathbb{R}^m$ at a point p of M seen as a submanifold with its tangent space at p where M is seen as a manifold (endowed with the differentiable structure naturally induced)?
2. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open neighborhoods. Let $\tilde{f} : U \rightarrow V$ be a smooth map such that $f := \tilde{f}|_{M \cap U} : M \cap U \rightarrow N \cap V$. Show that f is a smooth map between manifolds and that

$$df_p = \left(d\tilde{f}_p \right) \Big|_{T_p M} : T_p M \rightarrow T_{f(p)} N.$$

Exercise 3 (Tangent space of the torus). Let $p : \mathbb{R}^n \rightarrow \mathbb{T}^n$ be the quotient map and let M be a smooth manifold.

1. Using charts, show that \mathbb{T}^n is parallelizable.
2. Show that p is a diffeomorphism and that a map $f : \mathbb{T}^n \rightarrow M$ is smooth if and only if $f \circ p$ is smooth. Express the differential of $f : \mathbb{T}^n \rightarrow M$ by means of $f \circ p$.
3. We identify matrices $A \in \mathcal{M}_n(\mathbb{Z})$ with their induced maps $A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ (why is it well defined?). In which condition is $A \in \mathcal{M}_n(\mathbb{Z})$ a diffeomorphism of \mathbb{T}^n ?

Exercise 4 (Tangent space of a product). 1. Let M and N be two smooth manifolds, show that $T(M \times N)$ is diffeomorphic to $TM \times TN$.

2. Show that $T(\mathbb{S}^n) \times \mathbb{R}$ is diffeomorphic to $\mathbb{S}^n \times \mathbb{R}^{n+1}$. Deduce that $\mathbb{S}^n \times \mathbb{S}^1$ is parallelizable.

Exercise 5 (\mathbb{S}^3 is parallelizable). A Lie group is smooth manifold G endow with a group structure such that the multiplication $\mu : G \times G \rightarrow G$ and the inverse $\eta : G \rightarrow G$ are smooth maps.

1. Show that if G is a Lie group, then G parallelizable.

2. Show that $SU(2)$ is a Lie group diffeomorphic to \mathbb{S}^3

3. Deduce that \mathbb{S}^3 is parallelizable

Exercise 6 (Computation of a differential). Compute the differential of $\bar{F} : \mathbb{T}^2 \rightarrow \mathbb{S}^2$ defined as the quotient of the map from \mathbb{R}^2 to \mathbb{S}^2 :

$$F : (x, y) \mapsto (\cos(2\pi x) \cos(2\pi y), \cos(2\pi x) \sin(2\pi y), \sin(2\pi x)).$$

On which set is \bar{F} a local diffeomorphism? Is \bar{F} restricted to this domain a global diffeomorphism?

Exercise 7. 1. Show that every immersed n -manifold (that is a manifold of dimension $n > 0$) of \mathbb{R}^n is parallelizable.

2. Is it possible to immerse a compact n -manifold in \mathbb{R}^n ?

3. What is the minimal number of charts an atlas of \mathbb{S}^n can have?